Intro Video: Section 4.2
The Mean Value Theorem

Math F251X: Calculus 1

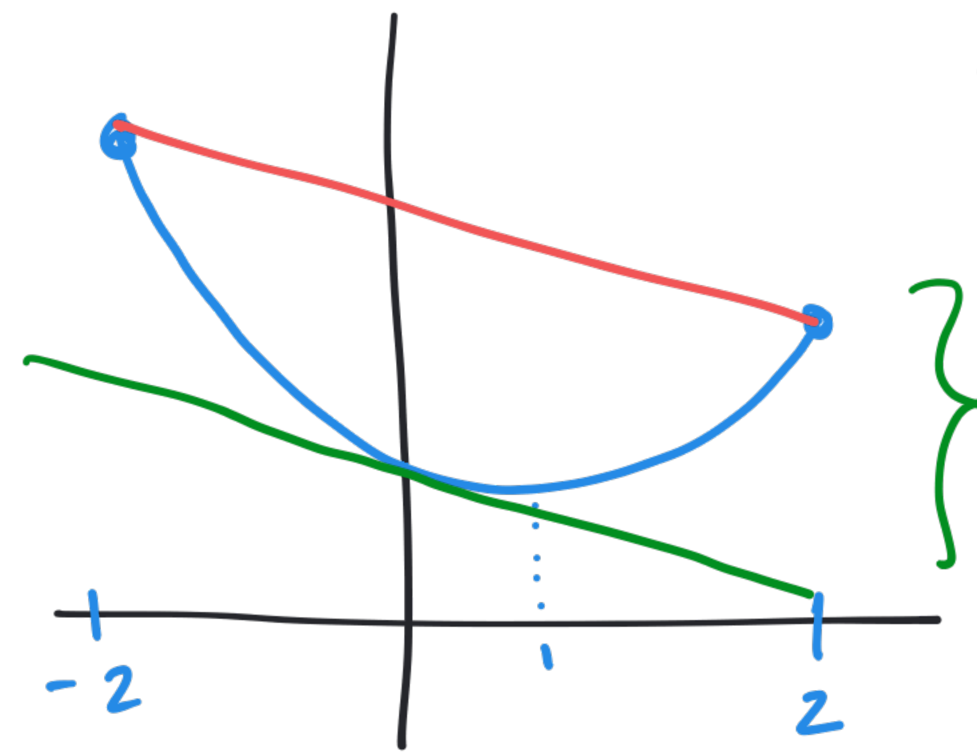
The Mean Value Theorem

$\underbrace{\text{Mean}}_{\downarrow}$

Average!

$f(x) = (x-1)^2 + 1$

Average rate of change = slope of secant line



Secant line is parallel to a tangent line!

$$= \frac{f(2) - f(-2)}{2 - (-2)}$$

$$= \frac{[(2-1)^2 + 1] - [(-2-1)^2 + 1]}{4} = \frac{2 - 10}{4} = \frac{-8}{4} = -2$$

Is there some x where $f'(x) = -2$? Well, $f'(x) = 2(x-1)$

So $f'(x) = -2 \Rightarrow -2 = 2(x-1) \Rightarrow -1 = x - 1 \Rightarrow x = 0$

The Mean Value Theorem says:

If f is

- continuous on $[a,b]$
- differentiable on (a,b)

  (this means the derivative exists for every x in (a,b))

f is "nice"

then there exists some c in (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

slope of tangent line at $(c, f(c))$

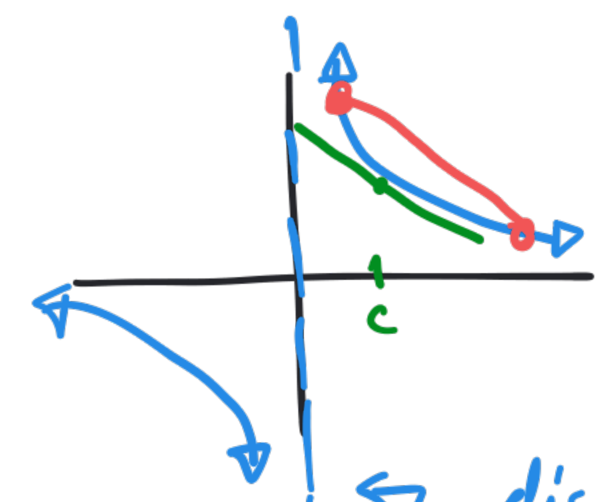
slope of secant line connecting $(b, f(b))$ and $(a, f(a))$

Example: Verify the Mean Value Theorem works for the function $f(x) = \frac{1}{x}$ on the interval $[1, 5]$.

Hypotheses: Is $f(x)$ continuous on $[1,5]$? Yes!

Does $f'(x)$ exist on $(1,5)$?

$f'(x) = -1x^{-2} = \frac{-1}{x^2}$ ← undefined only at $x=0$



← discontinuous at $x=0$

MVT Claims:

Slope of secant line $= \frac{f(b)-f(a)}{b-a} = \frac{\frac{1}{5} - \frac{1}{1}}{5-1} = \frac{\frac{1}{5} - \frac{5}{5}}{4} = \frac{-\frac{4}{5}}{4} = -\frac{1}{5}$

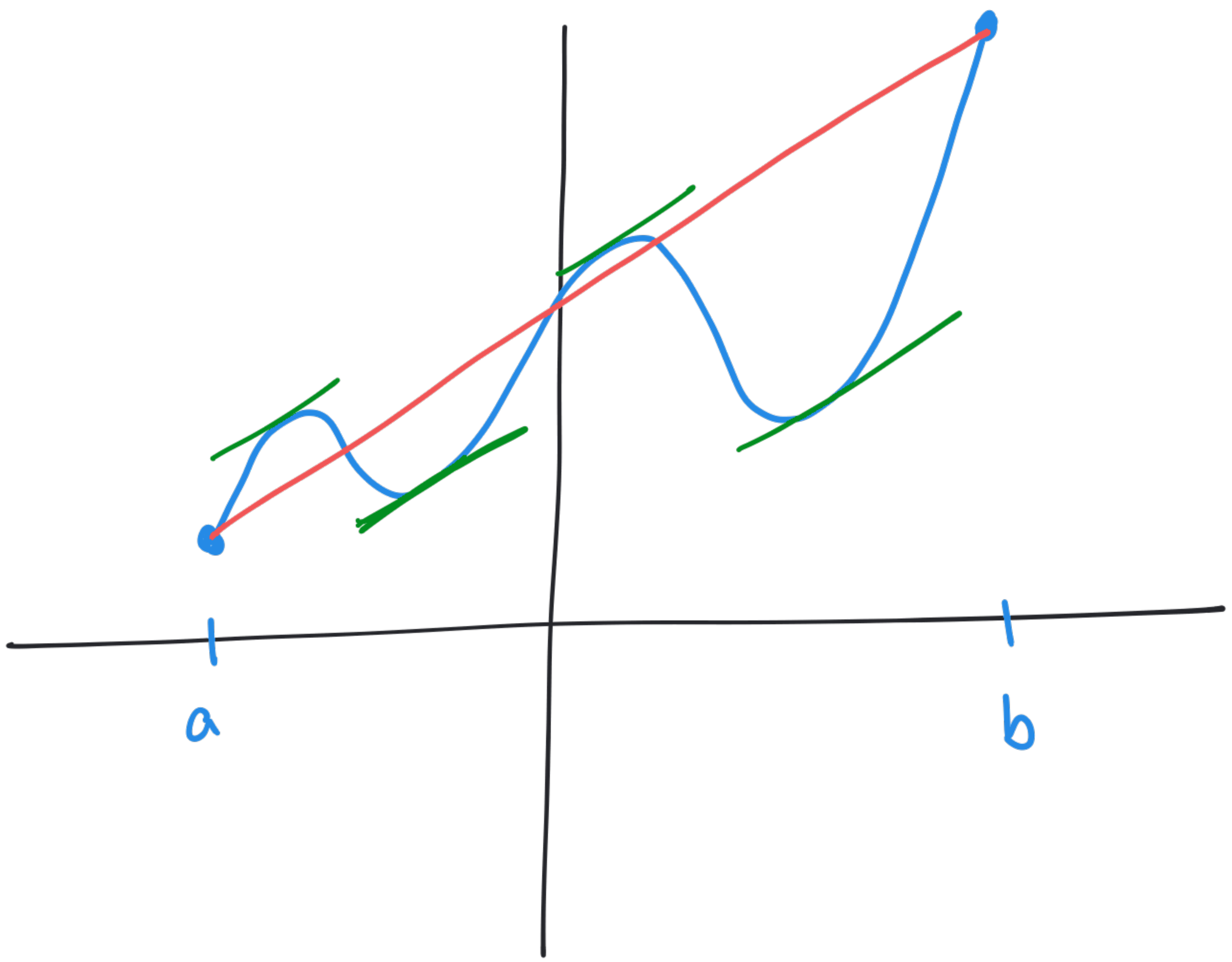
There exists some c in (a,b) such that

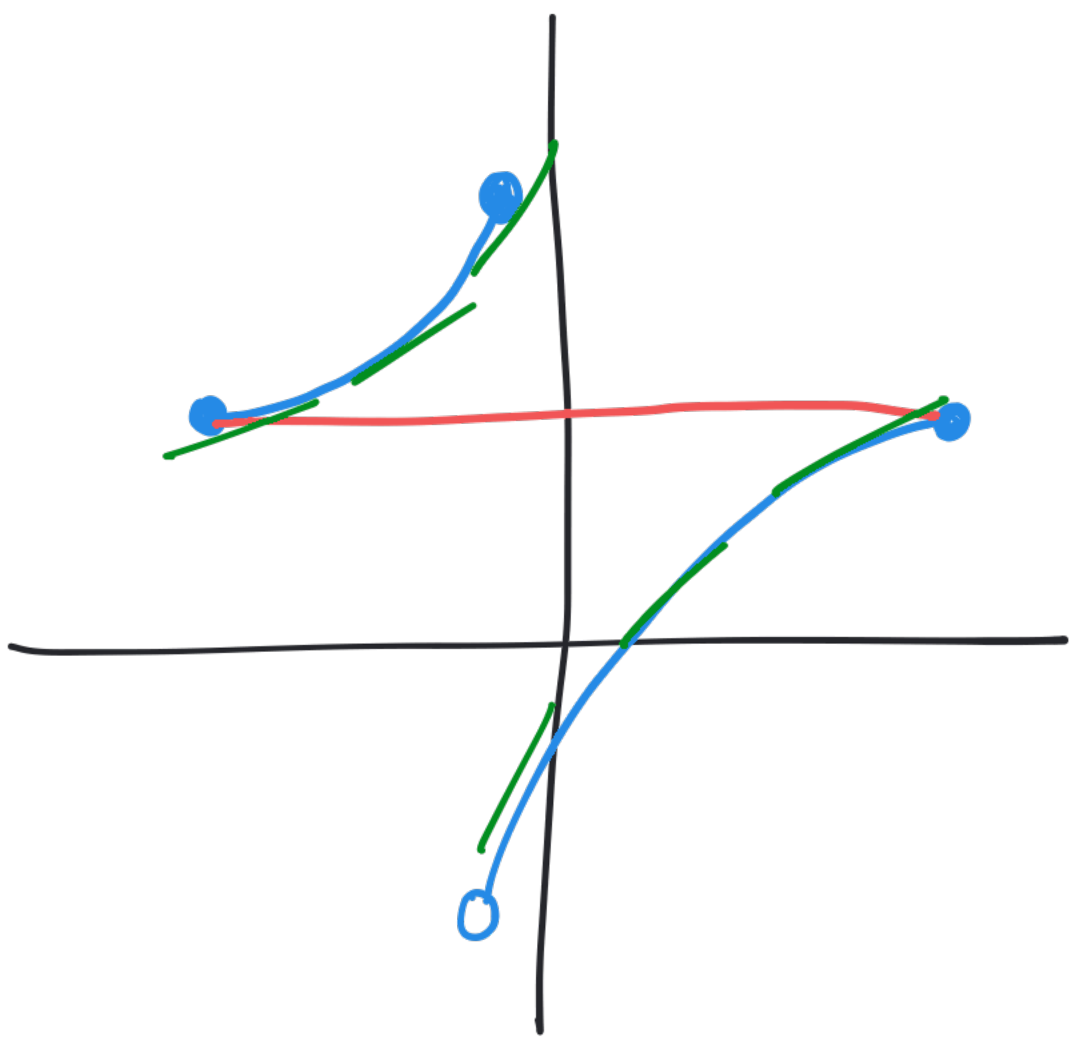
$f'(c) = -\frac{1}{5} \Rightarrow -\frac{1}{x^2} = -\frac{1}{5} \Rightarrow x^2 = 5 \Rightarrow \boxed{x = \sqrt{5}}$

or $x = -\sqrt{5}$

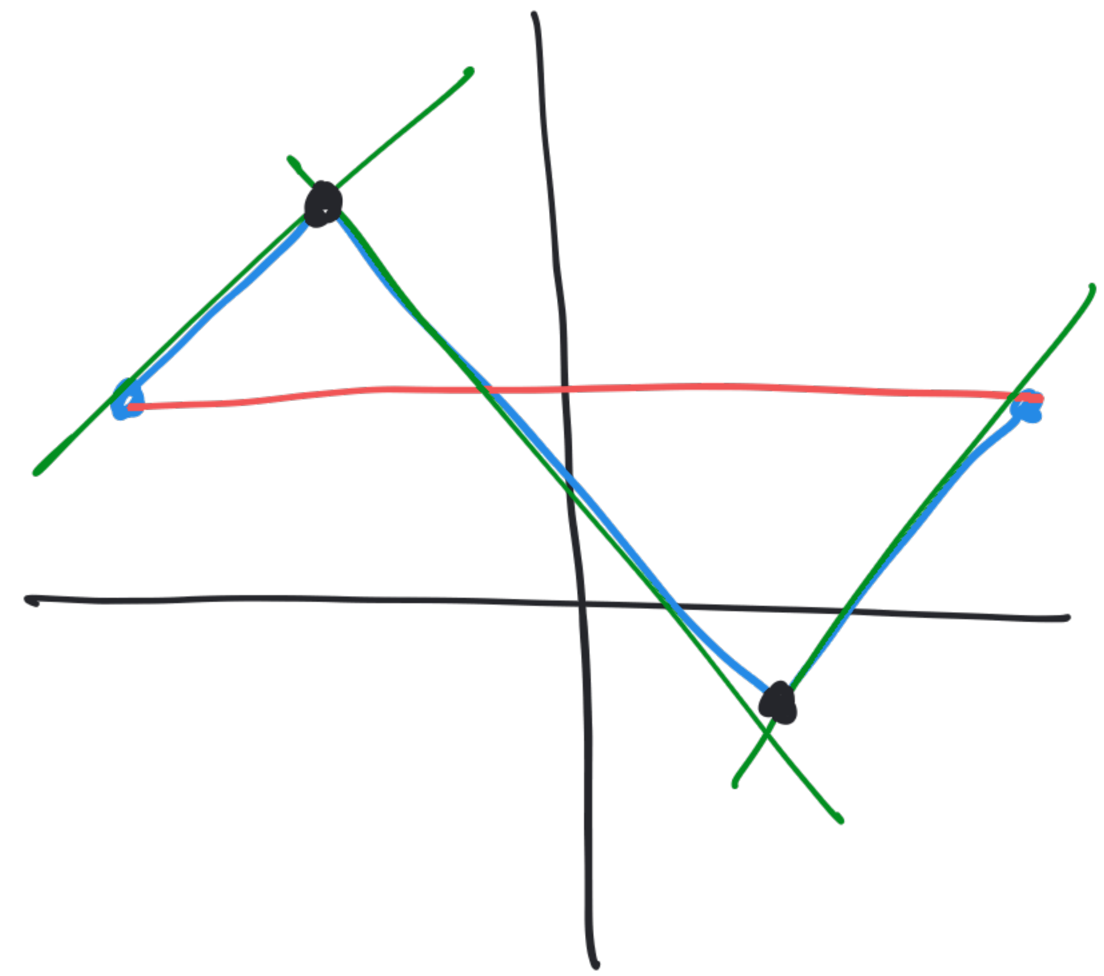
in $(1,5)$

The mean value theorem says at least one value exists. It does not say how many!





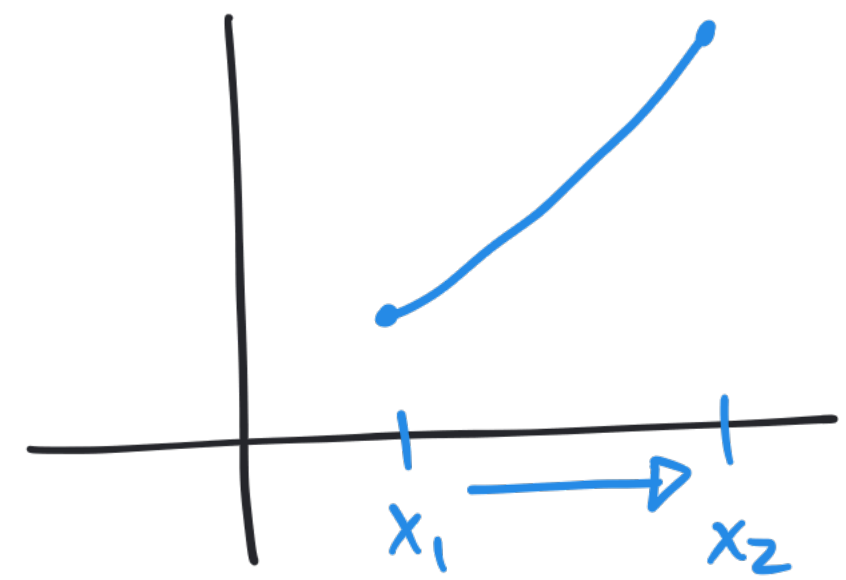
Mean value theorem conclusion may not hold if f is not continuous!



Mean value theorem conclusion may not hold if f is not differentiable!

Why do we care?

Def'n: A function is increasing if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$



Suppose we know $f'(x) > 0$ for all $x \in (x_1, x_2)$.

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) > 0 \Longrightarrow$$

$x_1 < x_2 \Longrightarrow x_2 - x_1 > 0$

$f(x_2) - f(x_1) > 0\,(x_2 - x_1) > 0$ (positive) $= 0$ So

$$f(x_2) - f(x_1) > 0 \Longrightarrow \boxed{f(x_2) > f(x_1)}$$

We just used the mean value theorem to show:

If $f'(x) > 0$ on (x_1, x_2) then $f(x_2) > f(x_1)$ on (x_1, x_2).

If the derivative is <u>positive</u> on an interval

then

the function is INCREASING on that interval!

Example: Does there exist a function f so that $f(0) = -1$, $f(2) = 4$, $f'(x) \leq 2$ for all x?

Consider f on $[0,2]$ and suppose f is continuous and differentiable.

MVT says: there exists some $c \in (0,2)$ such that

$$\frac{f(2) - f(0)}{2 - 0} = f'(c) \leq 2 \implies$$

$$4 - (-1) \leq 2(2) \implies 5 \leq 4$$

$\boxed{\text{No way!}}$

The only way this could happen is for either f to be discontinuous somewhere in $[0,2]$ or to be not differentiable somewhere in $(0,2)$. Otherwise, IMPOSSIBLE!